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- A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is invertible i.f.f. its matrix $[T]$ is invertible.

e.g. $T(\vec{x}) =$ " $\vec{x}$ rotated by θ."

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- this is invertible and $T^{-1}(\vec{x}) =$ " $\vec{x}$ rotated by $-\theta$ "

$$[T]^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$\boxed{\text{e.g}}$ $L \subset \mathbb{R}^n$ line through $\vec{0}$ and $\text{proj}_L : \mathbb{R}^n \to \mathbb{R}^n$

$\boxed{\text{Not Invertible}}$

Linear Algebra (so far)

- Equations of Interest: linear systems
- "Objects" of Interest: vectors in $\mathbb{R}^n$
- Operations of interest: vector addition, scaler multiplications
- Functions of interest: linear/matrix transformations. $T: \mathbb{R}^n \to \mathbb{R}^m$

Def - The span of vectors $\vec{v}, \ldots \vec{v}_K \in \mathbb{R}^n$ is the set span $(\vec{v}_1, \ldots, \vec{v}_K)$ of all linear combos of $\vec{v}_1, \ldots, \vec{v}_K$

$$\text{Span}(\vec{v}_1, \ldots, \vec{v}_K) = \{ c_1\vec{v}_1 + \ldots + c_K\vec{v}_K \in \mathbb{R}^n$$

IDK abt this

$\boxed{\text{E.g.}}$ $\vec{v} \in \mathbb{R}^n$ non $\vec{0}$

span $(\vec{v})$ = line through $\vec{0} + \vec{v}$

- span$(\vec{0}) = \{\vec{0}\}$ = "trivial subspace of $\mathbb{R}^n$"

- Span$(\vec{e}_1, \dots, \vec{e}_n) = \mathbb{R}^n$ = any vector $\in \mathbb{R}^n$

- $\underbrace{\vec{v}, \vec{w}}_{\text{Non } \emptyset \text{ and not multiples of each other}} \in \mathbb{R}^n$ span $(\vec{v}, \vec{w})$ - "plane" through $\vec{0}$, $\vec{v}, \vec{w}$

$\boxed{\cdot \text{ Always get } 0 \text{ in the span.}}$

Def - Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

- The image (or range) of T is the set

$$\operatorname{img}(T) = \{ T(\vec{x}) \in \mathbb{R}^m : \vec{x} \in \mathbb{R}^n \}$$

- The Kernal (or nullspace) of T is the set

$$\operatorname{Ker} T = \{ \vec{x} \in \mathbb{R}^n : T(\vec{x}) = \vec{0} \}$$

Ker (T) + image (T) are spans (of what?)

- Let $A = [T]$ + write $A = [\vec{v}_1, \dots, \vec{v}_n]$ column-wise. Then $\vec{y} \in$ image (T) means $\exists$ $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = A\vec{x} = \vec{y}$

$\boxed{\text{Example}}$

$T : \mathbb{R}^5 \to \mathbb{R}^4$ given by $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$

$x \in$ Kernel (T) i.f.f. $T(\vec{x}) = A\vec{x} = 0$

Need to solve: $[A \mid \vec{0}]$

A row equivalent to $\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

So: Kernel (T) = span $\left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Also Image (T) = Span $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \dots, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$ = span of cols of A

Possible Questions:

is $\vec{x} = \begin{bmatrix} \vdots \end{bmatrix} \in$ Kernel (T)? Ans: if $A\vec{x} \neq \vec{0}$ then no.

is $\vec{x} = \begin{bmatrix} 2 \\ 7 \\ 5 \\ 12 \end{bmatrix} \in$ Image(T)? Ans: solve $A\vec{x} = \begin{bmatrix} 2 \\ 7 \\ 5 \\ 12 \end{bmatrix}$